3.1 Derivatives of Polynomials and Exponentials

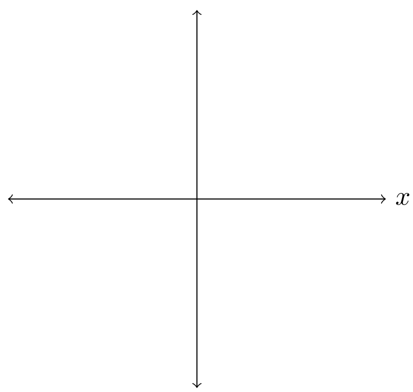
Learning Objectives: After completing this section, we should be able to

- find the derivative of a constant function using the definition of a derivative.
- derive the derivative of a power function with an integral exponent.
- derive the Constant Multiple Rule, the Sum Rule, and the Difference Rule.
- apply the general Power Rule to find the derivative of a power function with a real-valued exponent.

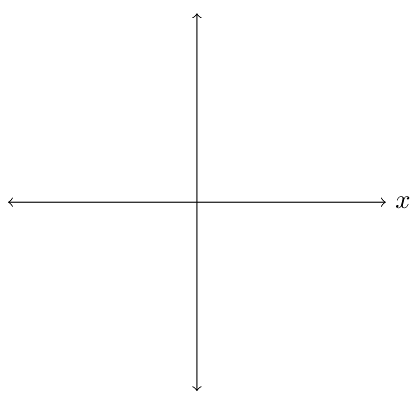
3.1.1 Polynomials

Evaluating $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ is tedious... but instructive! Let's establish some shortcuts, noting that all of them come from the definition.

- **Constant Rule:**

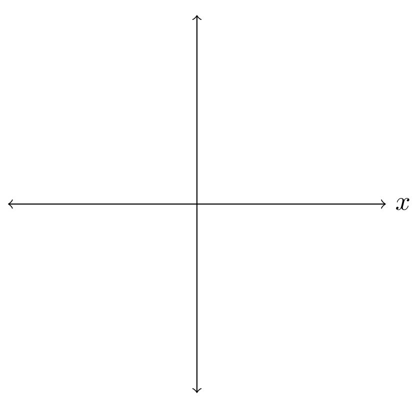


- **Power Rule:**

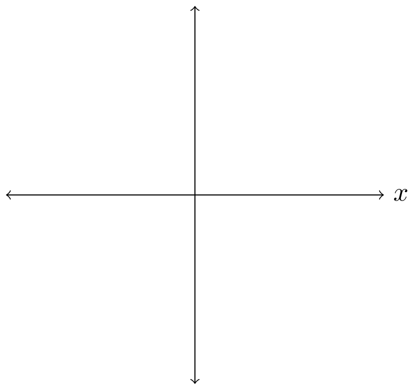


Power rule continued:

- **Constant Multiple:**



- **Sums:**



Example. Find the derivative of $f(x)$.

1. $f(x) = x^{12}$

2. $f(x) = 2x^{12}$

3. $f(x) = 2^{12}$

4. $f(x) = \frac{-7}{8}x^3$

You try!

5. $f(x) = 5x^3 + 3x + 1$

6. $f(x) = 3x^{-5}$

You try!

7. $f(x) = \pi x^{\pi}$

8. $f(x) = \frac{\pi}{2}\sqrt{x}$

9. $f(x) = \sqrt[m]{x^n}$

3.1.2 Exponential Functions

Exponential Functions Base e:

Example. Find the derivative of $f(x)$.

1. $f(x) = \frac{e^x}{4}$

2. $f(x) = e^3 x^4$

3. $f(x) = e^\pi + x^\pi + \pi e^x$

4. $f(x) = ee^x = e^{x+1}$

   You try!

5. $f(x) = 17e^x - x^{17}$

Example. Find all derivatives of $f(x) = x^3 - 3x^2 + 2$

You try!

Example. Find all derivatives of $f(x) = 3e^x + x$

3.2 Product and Quotient Rules

Learning Objectives: After completing this section, we should be able to

- derive and apply the Product Rule, and the Quotient Rule.

3.2.1 Product Rule

Question. *Is* $\frac{d}{dx}(f(x) \cdot g(x)) = f'(x)g'(x)$*?*

Theorem. *The product rule states* $\frac{d}{dx}(f(x) \cdot g(x)) =$

Example. Find the derivative of $h(x) = (3x^2 + 14x)(8e^x + 1)$.

You try!

Example. Find the derivative of $h(x) = (x^3 + 2)(2x^{-4} - x^{-1})$

You try!

Example. Find the derivative of $h(x) = 4x^3e^x$

Example. Find the derivative of $y = e^x(x^2+1)(3x-5)$.

You try!

Example. Evaluate $\frac{d}{dx}(xe^{2x})$.

3.2.2 Quotient Rule

Theorem. *The quotient rule states* $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) =$

Example. Find the derivative of $h(x) = \frac{x^4-8x^2}{x-1}$.

You try!

Example. Find the derivative of $y = \frac{e^x-2x}{1-xe^x}$.

Example. Find the derivative of $y = \frac{3x-e^x}{2}$.

Example. Find the derivative of $y = \frac{2x+e^x}{x}$.

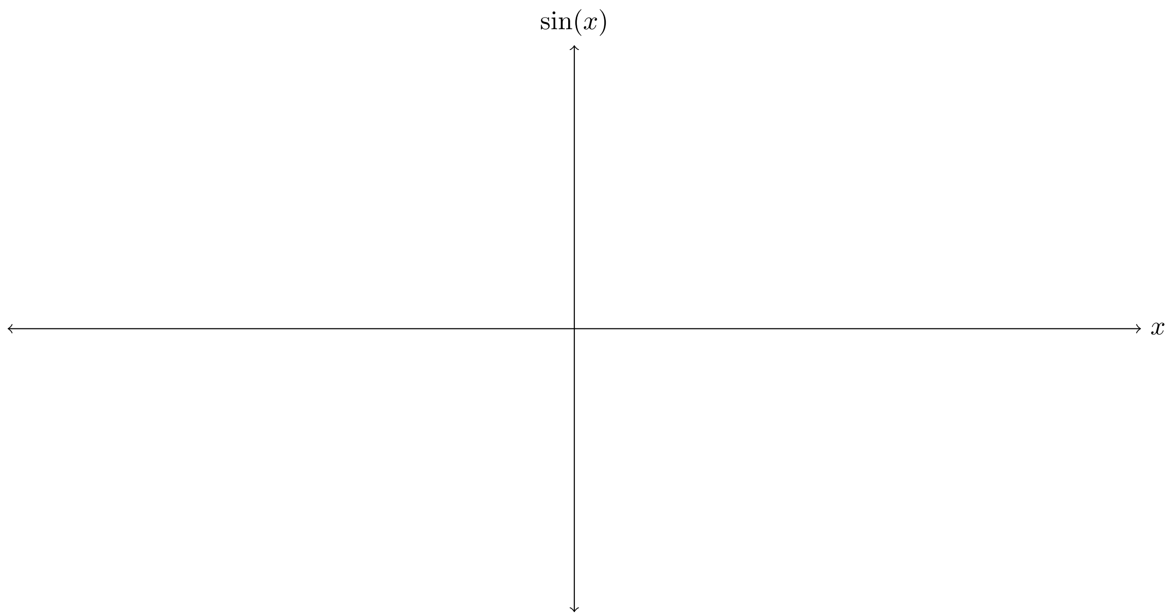
Example. Find the derivative of $y = \frac{x^\pi - \sqrt{x}}{x^3}$

3.3 Derivatives of Trigonometric Functions

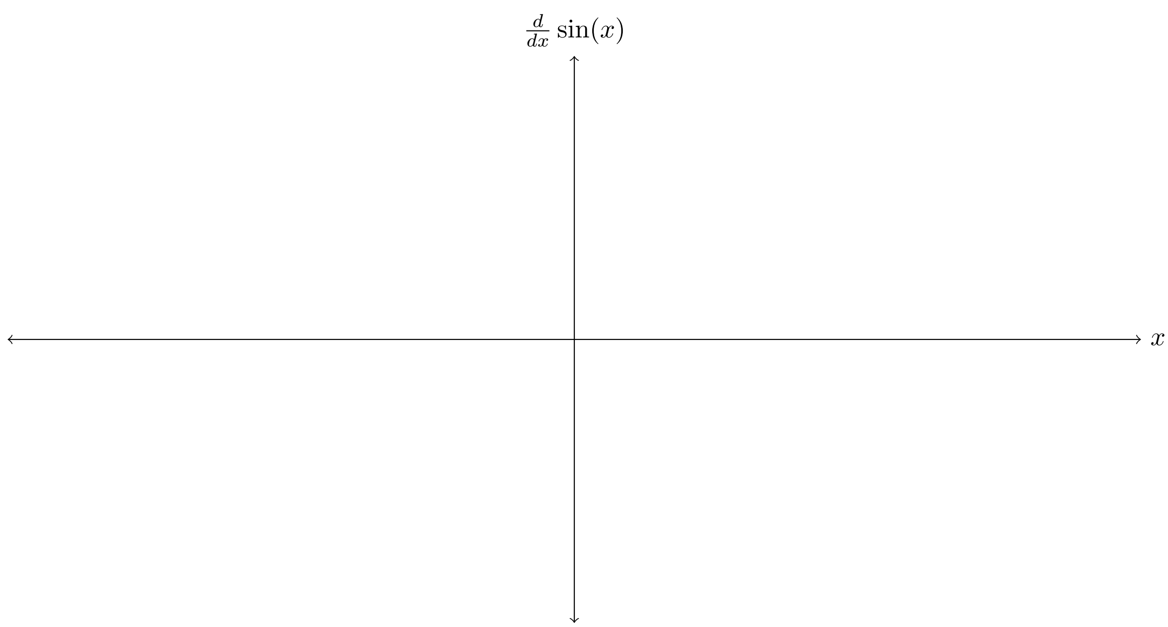
Learning Objectives: After completing this section, we should be able to

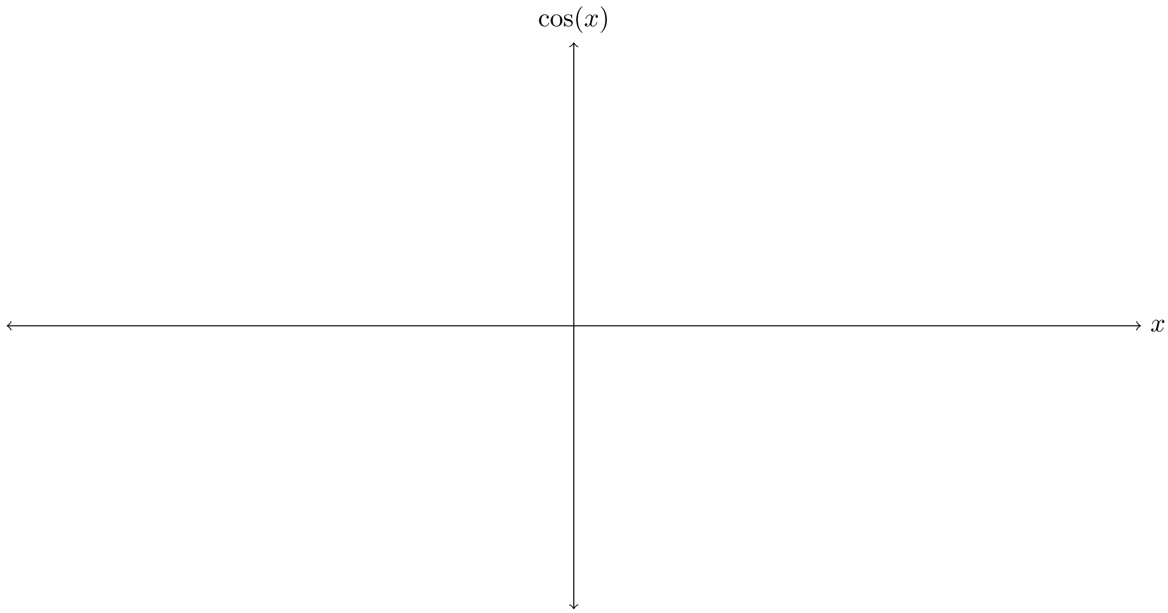
- derive and apply derivatives for trigonometric functions.

We are going to focus on derivatives of trigonometry functions.

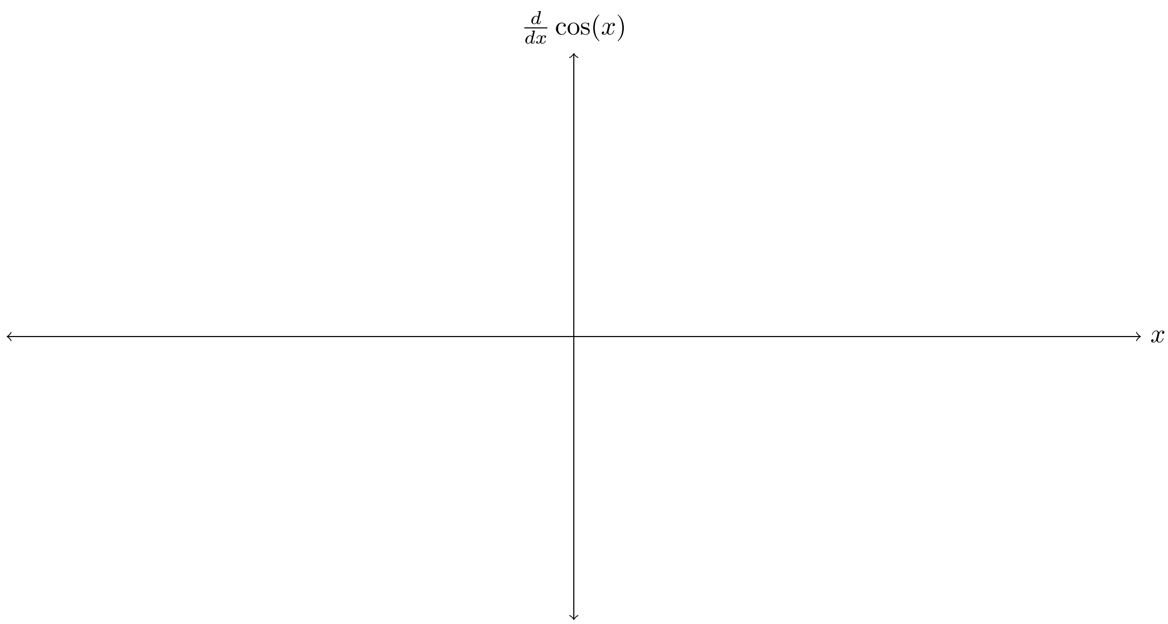


Let's graph it's derivative





Let's graph it's derivative



Example. Find the derivative of $f(x) = \tan(x)$.

Using the quotient rule, we can find the derivatives of $\csc(x)$, $\cot(x)$, and $\sec(x)$. Simplify!

$f(x)$	$f'(x)$
$\sin(x)$	
$\cos(x)$	
$\tan(x)$	
$\csc(x)$	
$\sec(x)$	
$\cot(x)$	

You try!

Example. Find the derivative of $f(x)$ for the following problems.

1. $f(x) = e^x \sin(x)$.

2. $f(x) = \frac{x \tan(x)}{1+\cos(x)}$.

3. $f(x) = \sec(x) \tan(x)$.

4. $f(x) = \cot(x) \cos(x)$.

3.4 Chain Rule

Learning Objectives: After completing this section, we should be able to

- apply the chain rule to obtain the derivative of a composite function.
- apply the chain rule to obtain the derivative of a power function and an exponential function.

Example. Let's try to find the derivative of $y = (3x^2 + x)^2$.

Theorem. *The chain rule states* $\dfrac{d}{dx} f(g(x)) =$

Example. Again, consider the function $y = (3x^2 + x)^2$.

Example. $y = (4x^2 + x^{-2})^4$.

Example. $y = \cos(x^2 - 1)$.

You try!

Example. $y = \tan(5x^2 + 2x)$.

We can combine rules.

Example. $y = \cos^3(x^2 - 1)$.

You try!

Example. $y = \sqrt{\sin(3x)}$.

You try!

Example. $y = \left(\dfrac{\sin(x)}{1 + \cos(x)}\right)^5$.

Recall we have the derivative of $y = e^{2x}$ is $y' = 2e^{2x}$. Let's think of it with the chain rule.

You try!

Example. $y = e^{\left(4x^2 - 5x + 1\right)}$.

You try!

Example. $y = 2e^{\left(3\cos^2(x^4)\right)}$.

Forgot the quotient rule? No problem!

When do we need the quotient rule?

1. $y = \dfrac{x^4 + e^x}{5}$.

2. $y = \dfrac{5}{x^4 + e^x}$.

3. $y = \dfrac{x^4 + e^x}{x - 1}$.

3.6 Derivatives of Logarithms and Inverse Trigonometric Functions

Learning Objectives: After completing this section, we should be able to

- define the general logarithmic functions
- derive the derivative of logarithmic functions.
- derive the derivatives of all inverse trigonometric functions.

3.6.1 Logarithmic Functions

Key properties of logarithms:

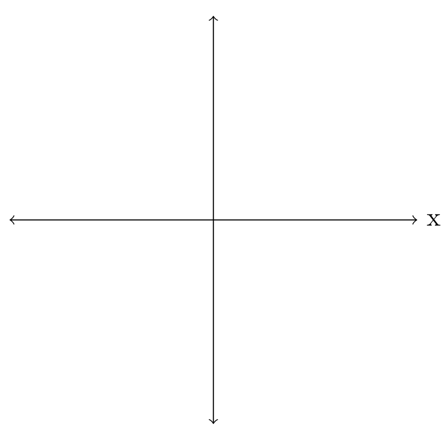
1. $\log(b^x) =$

2. $\log(x \cdot y) =$

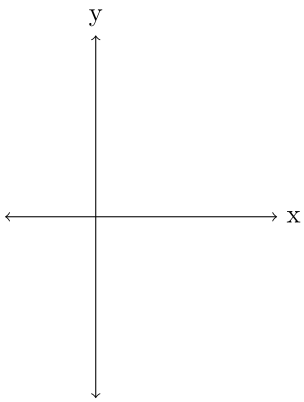
3. $\log\left(\frac{x}{y}\right) =$

4. $\log(x + y) =$

What is $y = \log_e(x) = \ln(x)$?



Other Bases:


y
x

Logarithms and exponentials have an inverse relationship.

Question. *How do we take the derivative of a logarithm?*

Let's practice!

1. $y = 2^x$

2. $y = \pi^x$

3. $y = 3^{x^2+2x+1}$

4. $y = 8 \cdot 3^x$

5. $y = 4^{-x}$

6. $y = 4\ln(3x)$

7. $y = \ln(12x^5) + 12x^5$

8. $y = \ln(\tan(x))$

9. $y = \log_5(3x)$

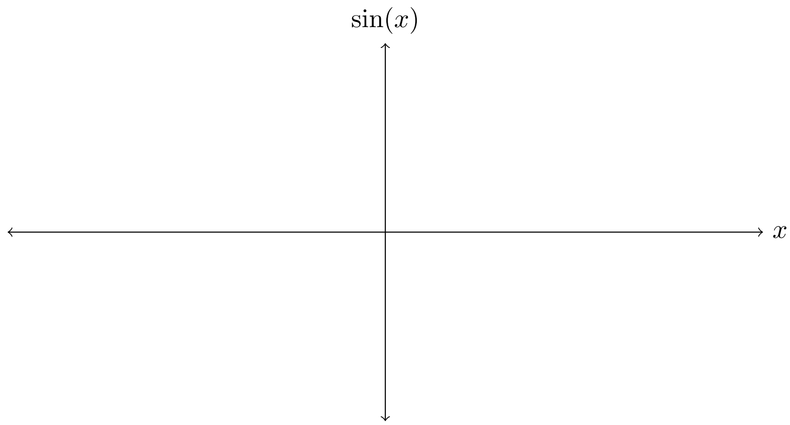
10. $y = \log_b(\tan(x))$

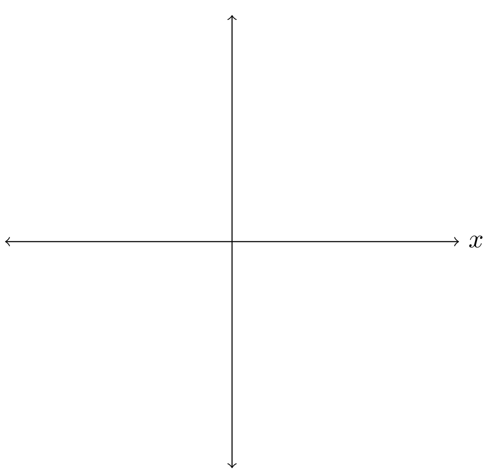
11. $y = \log_2(e^x)$

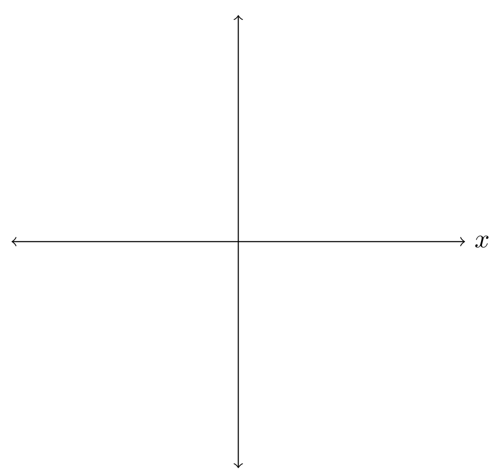
3.6.2 Inverse Trigonometric Functions

Consider $y = \sin^{-1}(x) =$

Let's sketch a graph of $\sin(x)$







Example. Compute $\arcsin\left(\dfrac{\sqrt{3}}{2}\right)$.

Example. Compute $\arcsin\left(\sin\left(\dfrac{3\pi}{4}\right)\right) =$

Example. Compute $\arccos\left(\frac{\sqrt{3}}{2}\right) =$

Example. Compute $\arctan\left(-\frac{1}{\sqrt{3}}\right) =$

How do we find derivatives of inverse trig?

Example. Let $y = \arctan(x) =$

Example. Let $y = 10\arctan(4x^2)$. Find y'.

You try!

Example. Find a formula for $\dfrac{d}{dx}\arcsin(x)$.

You try!

Example. Find a formula for $\dfrac{d}{dx}\arccos(x)$.

3.6.3 Summary

- $\frac{d}{dx}e^{f(x)} =$

- $\frac{d}{dx}b^{f(x)} =$

- $\frac{d}{dx}\ln(f(x)) =$

- $\frac{d}{dx}\log_b(f(x)) =$

- $\frac{d}{dx}\arccos(x) =$

- $\frac{d}{dx}\arcsin(x) =$

- $\frac{d}{dx}\arctan(x) =$

3.9 Related Rates

Learning Objectives: After completing this section, we should be able to

- solve related rates problems in various real-life situations.

Related rates are all about how multiple rates of change are connected. For example, if I drive north at 15 mph and you drive south at 15 mph, the distance between us is increasing at a rate of 30 mph.
Tips:

- Interpret the derivatives

- You may need to come up

- Don't mix up

Example. Two boats leave a port at 12:00pm. One travels West at 20mph and the other South at 15mph. How fast are they moving away from each other at 1:30pm?

Example Continued:

Example. A ladder 13 feet long leans against a building. The base is pulled away from the wall at a rate of $\frac{1}{10} \frac{\text{ft}}{\text{sec}}$. How fast is the top moving down when the top is 12 feet above ground?

You try!

Example. Suppose you are blowing up a spherical balloon at a rate of 3 $\frac{\text{cm}^3}{\text{s}}$. How fast is the radius of the balloon changing when $r = 5$ cm?

Example. John Nader is quickly walking north on Broadhollow Road at 5 feet per second while watching Rambo the Ram trotting west on Melville Road at 10 feet per second. At the moment, Rambo is at the intersection of Melville and Broadhollow and John Nader is 55 feet away from the same intersection.

After three seconds, at what rate is the distance between Marvin and Blaster increasing? Remember to use accurate units!